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Regular p -Groups, III

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Let G be a finite p -group, and let $N \triangleleft G$. We say that N is *regularly embedded* in G if, for all $a \in G$, $b \in N$,

$$(ab)^p = a^p b^p u_1^p \cdots u_k^p, \quad u_i \in H', \quad H = \langle a, b \rangle. \quad (1)$$

This notion is investigated in the first section of this article, and we obtain some analogues and extensions of well-known properties of regular p -groups. We also introduce the subgroup $\mathcal{U}_n(N; G)$, a “relativization” of $\mathcal{U}_n(N)$, and relate it to regular embedding. In the second section we investigate *totally regular* groups, these being the groups K that are regularly embedded in G whenever they are normal in G . For example, a group with $|K : \mathcal{U}_1(K)| \leq p^{p-2}$ is totally regular. Total regularity is essentially a condition on the automorphisms of K , so several recent papers constructing p -groups with “small” automorphism group yield examples of totally regular groups. These show that total regularity is not inherited by either subgroups or factor groups. Under the assumptions that all sections of K are totally regular, we derive severe restrictions on the structure of K , and probably more is true. Finally, in the last section we discuss briefly the notions of *marginal series* and *marginal products* of varieties, a natural generalization of some of the preceding concepts. We also believe that the series of subgroups $\mathcal{U}_{(n)}(G)$ and $W_i(G)$, defined respectively in the Remark to Theorem 4 and at the beginning of Section 3, are of interest.

Notation and terminology. If V is a variety of groups, i.e., the class of all groups G satisfying $v_\alpha(a_1, \dots, a_n) = 1$, $a_i \in G$, for some set of words $\{v_\alpha\}$, we say that V is *defined* by the set of words $\{v_\alpha\}$. For an arbitrary group G , the *verbal subgroup* $V(G)$ is the subgroup generated by all elements $v_\alpha(a_1, \dots, a_n)$, for all a and all $a_i \in G$, and the *marginal subgroup* $M_V(G)$ is the set of elements $z \in G$ such that $v_\alpha(a_1, \dots, a_i z, \dots, a_n) = v_\alpha(a_1, \dots, a_i, \dots, a_n)$ for all α , $a_i \in G$, and all $i = 1, \dots, n$. In particular, $M(G)$ is the marginal subgroup corresponding to the word X^p . In Sections 1 and 2 all groups considered are finite p -groups. For such a group G , $\Omega_i(G)$ and $\mathcal{U}_i(G)$ denote, respectively,

the subgroups generated by all elements of order at most p^i and by all p^i th powers in G . The center, Frattini subgroup, derived group, n th term of the lower central series of G are denoted, respectively, by $Z(G)$, $\Phi(G)$, G' , G_n , while $A(G)$ is the automorphism group of G . An automorphism of G of p -power order is called a p -automorphism, and $d(G)$, $\text{cl } G$, $\exp G$ denote the minimal number of generators and the class and exponent of G . Finally, for $p = 2$ regularity is equivalent to commutativity, hence regular embedding is the same as being central, and the only totally regular 2-groups are the groups of order 1 and 2. Hence we assume in Sections 1 and 2 that p is odd. The maximal order of an abelian subgroup of G is $p^{m(G)}$.

Standard references for regular p -groups are [3, 6.III.10] and for varieties [9].

1. REGULAR EMBEDDING

PROPOSITION 1. *Let $N \triangle G$. If $N \subseteq Z_{p-1}(G)$, or $N \subseteq M(G)$, or $|N| \leq p^{p-1}$, then N is regularly embedded in G .*

The proof is immediate.

PROPOSITION 2. *Let $N \triangle G$. Then N is regularly embedded in G if and only if $\langle a, N \rangle$ is regular for all $a \in G$.*

Proof. Let N be regularly embedded. We use induction on G . Let $a \in G$. We may assume $G = \langle a, N \rangle$. Let $1 \neq K \triangle G$. By induction G/K is regular. Also, if H is a proper subgroup of G , then $H \cap N$ is regularly embedded in H and $H/H \cap N$ is cyclic, so H is regular. Thus, if G is irregular, it is a so-called minimal irregular group. By [8, Theorem 2], G' has exponent p , so (1) implies.

$$(ab)^p = a^p b^p, \quad a \in G, b \in N. \quad (2)$$

Now G/N is cyclic, so $G' \subseteq N$. Also G is 2-generator, so we can find elements a, b with $b \in N$ and $G = \langle a, b \rangle$. Let $x, y \in G$, and write

$$x = a^i b^j u, \quad y = a^k b^l v, \quad u, v \in G'.$$

By [8, Theorem 2] p th powers are central in G , so, for some $w \in G'$:

$$\begin{aligned} (xy)^p &= (a^{i+k} b^{j+l} w)^p = (a^{i+k} b^{j+l})^p = a^{(i+k)p} b^{(j+l)p} \\ &= a^{ip} b^{jp} a^{kp} b^{lp} = (a^i b^j)^p (a^k b^l)^p = x^p y^p \end{aligned}$$

and thus G is regular.

The other direction is trivial.

PROPOSITION 3. *Let $N \triangle G$. Then N is regularly embeded in G if and only if for each $a \in G$, the semi-direct product $N\langle a \rangle$ is regular, where a is the automorphism induced on N by a .*

Proof. We show that $\langle N, a \rangle$ and $N\langle a \rangle$ are either both regular or both irregular. Let p^n be the order of a . Assume that $N\langle a \rangle$ is regular, then so is the direct product $N\langle a \rangle \times \langle \beta \rangle$, where β has order p^n . Then $a\beta$ induces a on N and is of order p^n , so $\langle N, a \rangle$ is a homomorphic image of $\langle N, a\beta \rangle$ and is therefore regular. If $\langle N, a \rangle$ is regular, regularity of $N\langle a \rangle$ follows in the same way.

Let $N \triangle G$ and $C = C_G(N)$. Then Proposition 3 implies that N is regularly embedded in G if and only if N is regularly embedded in the semi-direct product $N \cdot G/C$.

THEOREM 4. *Let $N, M \triangle G$ with $[N, M]$ regularly embedded. Then*

$$[\mathcal{U}_n(N), \mathcal{U}_m(M)] = \mathcal{U}_{n+m}([N, M]). \quad (3)$$

Proof. It suffices to consider the cases in which $m=0$ or $n=0$. We discuss only the case $m=0$, the other one being similar to it. To show that $[\mathcal{U}_n(N), M] \subseteq \mathcal{U}_n([N, M])$, we assume that $\mathcal{U}_n([N, M]) = 1$ and take $a \in N$, $b \in M$. Then $\langle a, a^b \rangle = \langle a, [a, b] \rangle$ is regular and $(a^{-1}b^{-1}ab)^{p^n} = 1$ so, by properties of regular groups,

$$a^{p^n} = (b^{-1}ab)^{p^n} = b^{-1}a^{p^n}b, \quad [a^{p^n}, b] = 1$$

so $[\mathcal{U}_n(N), M] = 1$. This proves $[\mathcal{U}_n(N), M] \subseteq \mathcal{U}_n([N, M])$. Reversing the argument shows that, if $[\mathcal{U}_n(N), M] = 1$, then $[a, b]^{p^n} = 1$, so $\mathcal{U}_n([N, M]) = 1$, since $[N, M]$ is regular. Thus $\mathcal{U}_n([N, M]) \subseteq [\mathcal{U}_n(N), M]$.

Remark. Let us define $\mathcal{U}_{(1)}(G) = \mathcal{U}_1(G)$, $\mathcal{U}_{(i)}(G) = \mathcal{U}_1(\mathcal{U}_{(i-1)}(G))$; then $\mathcal{U}_n(G) \leq \mathcal{U}_{(n)}(G)$ and usually the inclusion is strict. Theorem 4 shows that, for $M, N \triangle G$ with N regularly embedded, we have

$$[N, \mathcal{U}_m(M)] = [N, \mathcal{U}_{(m)}(M)] \quad (= \mathcal{U}_m([N, M])).$$

DEFINITION. Let $N \triangleleft G$. We define

$$\mathcal{U}_n(N; G) = \langle (ab)^{p^n} a^{-p^n} \mid a \in G, b \in N \rangle.$$

Obviously

$$\mathcal{U}_n(N) \subseteq \mathcal{U}_n(N; G) \subseteq \mathcal{U}_n(G) \cap N.$$

THEOREM 5. *Let N be regularly embedded in G . Then $\mathcal{U}_n(N; G) = \mathcal{U}_n(N)$. Moreover, for any given element $a_0 \in G$,*

$$\mathcal{U}_n(N; G) = \{(a_0 b)^{p^n} a_0^{-p^n} \mid b \in N\}.$$

Proof. Working $\text{mod}(\mathcal{U}_n(N))$, we may assume that N has exponent p^n . Then $\langle a, b \rangle$ is regular, for $a \in G$, $b \in N$, and $b^{p^n} = 1$, so $(ab)^{p^n} a^{-p^n} = 1$. Thus $\mathcal{U}_n(N; G) \subseteq \mathcal{U}_n(N)$.

Now fix $a_0 \in G$, and consider the map $b \rightarrow (a_0 b)^{p^n} a_0^{-p^n}$ from N to $\mathcal{U}_n(N)$. If $b, c \in N$, then b and c have the same image if and only if $(a_0 b)^{p^n} = (a_0 c)^{p^n} = (a_0 b \cdot b^{-1}c)^{p^n}$, i.e., since $\langle a_0, b, c \rangle$ is regular, if and only if $(b^{-1}c)^{p^n} = 1$. Thus our map is 1-1 from $N/\Omega_n(N)$ into $\mathcal{U}_n(N)$. Since $|N : \Omega_n(N)| = |\mathcal{U}_n(N)|$, the map is onto $\mathcal{U}_n(N) = \mathcal{U}_n(N; G)$.

THEOREM 6. *If $N \triangle G$ and $|N : \mathcal{U}_1(N; G)| \leq p^{p-2}$, then N is regularly embedded in G .*

Proof. We first show that $\mathcal{U}_1(N; G) = \mathcal{U}_1(N)$. Suppose not, and find a $K \triangle N$ such that $\mathcal{U}_1(N) \subseteq K \subseteq \mathcal{U}_1(N; G)$, $|\mathcal{U}_1(N; G) : K| = p$. Then in G/K the normal subgroup N/K is of exponent p and order $\leq p^{p-1}$, so regularly embedded, and by Theorem 5 $\mathcal{U}_1(N; G) \subseteq K$, a contradiction.

Now let $a \in G$ and $H = \langle a, N \rangle$. Then $H/\mathcal{U}_1(H)$ is generated by the normal subgroup $N\mathcal{U}_1(H)/\mathcal{U}_1(H)$, of order p^{p-2} at most, and the element $a\mathcal{U}_1(H)$, of order p at most. Thus $|H : \mathcal{U}_1(H)| \leq p^{p-1}$ and H is regular.

COROLLARY 7. *If $N \triangle G$ and $|N : \mathcal{U}_1(N)| \leq p^{p-2}$, then N is regularly embedded in G .*

THEOREM 8. *Let $N \triangle G$. If N does not contain any normal subgroup of G of exponent p and order p^{p-1} , then N is regularly embedded in G .*

Proof. Let K be a maximal subgroup of N which is normal in G . By induction, K is regularly embedded in G . Since N/K is cyclic, N is regular. Now our assumption is equivalent to $|\Omega_1(N)| = |N : \mathcal{U}_1(N)| \leq p^{p-2}$ and we apply the previous result.

A normal subgroup of order p^{p-1} and exponent p , being regularly embedded, is always contained in $M(G)$. Therefore the assumption of Theorem 8 is equivalent to “ $|N \cap M(G)| \leq p^{p-2}$ ” and also to “ $|N \cap \Omega_1(Z_{p-1}(G))| \leq p^{p-2}$,” “no characteristic subgroup of exponent p of N has order p^{p-1} or more,” etc.

2. TOTALLY REGULAR GROUPS

DEFINITION. A group K is *totally regular* if, whenever $K \triangle G$, K is regularly embedded in G .

Thus groups satisfying the equivalent conditions $|K : \mathcal{U}_1(K)| \leq p^{p-2}$ or $|\Omega_1(K)| \leq p^{p-2}$ are totally regular, as are groups of order p^{p-1} or less. By

Proposition 3, K is totally regular if and only if the semi-direct products $K\langle\alpha\rangle$ are regular, for all p -automorphisms α of K .

THEOREM 9. *An abelian group K is totally regular if and only if either $|K : \mathcal{U}_1(K)| \leq p^{p-2}$ or $|K| = p^{p-1}$.*

Proof. If $K = L \times M$ is totally regular, so are L and M , by the remarks preceding this proposition. An abelian group K which does not satisfy either of the conditions of Theorem 9 has a direct factor L which is either elementary abelian of order p^p or a direct product of $p-1$ cyclic groups, not all of order p . In the first case L is a maximal subgroup of the wreath product of two groups of order p , and is thus not totally regular. In the second case, let $L = \langle a_1 \rangle \times \cdots \times \langle a_{p-1} \rangle$, with non-increasing orders of the a_i 's. Let a_p be an element of order p in a_1 , and let α be the automorphism of L defined by $a_i \rightarrow a_i a_{i+1}$, $i = 1, \dots, p-1$. The subgroup $M = \langle a_2^p, \dots, a_{p-1}^p \rangle$ is α -invariant. Let $\bar{L} = L/M$. On \bar{L} we have $\alpha^p = 1$ and $\bar{L}\langle\alpha\rangle$ is a group of class p with an abelian maximal subgroup \bar{L} , so $\bar{L}\langle\alpha\rangle$, and with it $L\langle\alpha\rangle$, is irregular [6, III.10.10], and L is not totally regular.

We can now show that total regularity is not inherited by subgroups or factor-groups. Let G be a p -group satisfying $G' = Z(G)$ and such that all p -automorphisms of G are central (i.e. induce the identity on $G/Z(G)$ and (therefore) on G'). Then the semi-direct product of G by the p -Sylow subgroup of $A(G)$ is of class 2, so G is certainly totally regular. Such groups G are constructed in [4, 5, 7] (in [4], take $(|K|, p) = 1$). In particular, the groups of [4] satisfy $|G : G'| = p^n$, $|G'| = p^{(n)}$, where n can be any integer ≥ 8 , while those of [5] satisfy $|G : G'| = |G'| = p^n$ for any $n \geq 4$, so in both cases neither G' nor G/G' is totally regular, if n is large enough. For the groups of [7] one has $|G : G'| = |G'| = p^4$, so these provide similar examples for $p = 3$, while for $p = 5$, G' and G/G' are totally regular, but the subgroups denoted in [7] by A and B are not.

PROPOSITION 10. *Let K be a 3-group. If either each subgroup, or each factor-group of K , is totally regular, then K is cyclic or of order 9.*

Proof. Let K be non-abelian. Then K has an elementary abelian normal subgroup of order 9. If each subgroup of K is totally regular, then N is a maximal abelian subgroup of K , by Theorem 9, and then $|K| \leq 27$. Now both non-abelian groups of order 27 are maximal subgroups of the irregular wreath product of two groups of order 3, so they are not totally regular. Thus K is abelian, and by Theorem 9 it is cyclic of order 9. Next, let each factor-group of K be totally regular for non-abelian K . Then Proposition 9 implies $|K : K'| = 9$. Let $N \triangle K$, $N \subseteq K'$, $|K' : N| = 3$. Then K/N is non-abelian of order 27, the same contradiction as before.

For larger p , we first discuss some groups of order p^p . By Corollary 7, we may assume $|\mathcal{U}_1(G)| \leq p$.

THEOREM 11. *Assume both $|G'| = p$ and $|\mathcal{U}_1(G)| \leq p$. We divide such groups into five types:*

- (a) $\text{Exp } G = p$.
- (b) $\text{Exp } G = p^2, \exp Z(G) = p, \mathcal{U}_1(G) = G'$.
- (c) $\text{Exp } G = p^2, \exp Z(G) = p, \mathcal{U}_1(G) \neq G'$.
- (d) $\text{Exp } G = p^2, \exp Z(G) = p^2, \mathcal{U}_1(G) = G'$.
- (e) $\text{Exp } G = p^2, \exp Z(G) = p^2, \mathcal{U}_1(G) \neq G'$.

Then G is a central product of groups from the following list: cyclic of order p or p^2 , non-abelian of order p^3 and a certain non-abelian group of order p^4 . Moreover, a complete set of isomorphism invariants for G is its type, order and the order of $Z(G)$. Finally, if $|G| = p^p$, G is totally regular if and only if it is of type (c), (d), or (e).

Proof. In the usual way, we consider $G/\Phi(G)$ and G' as vector space over $GF(p)$, and the commutator map $(a, b) \rightarrow [a, b]$ as an alternating bilinear form on $G/\Phi(G)$ (see, e.g., [6, III.13.7]). The radical of the form is $Z(G)/\Phi(G)$, and we can find a subgroup H such that $H/\Phi(G)$ is a subspace complementary to $Z(G)/\Phi(G)$. In cases a, d, e we can choose H to be of exponent p , because in these cases either $G = \Omega_1(G)$, or $|G : \Omega_1(G)| = p$ and $Z(G) \not\subseteq \Omega_1(G)$, so $G = Z(G)\Omega_1(G)$. On $H/\Phi(G)$, the alternating form is non-degenerate, and we can choose a symplectic basis $\langle a_i, b_i, i = 1, \dots, n \rangle$ for $H/\Phi(G)$. If $\exp H \neq p$, let $H_1 = H \cap \Omega_1(G)$, then $H_1/\Phi(G)$ is a maximal subspace of $H/\Phi(G)$, and the basis can be chosen to lie in H_1 except for, say, a_1 . Then G is the central product of $Z(G)$ and the subgroups $\langle a_i, b_i \rangle$. The last are non-abelian, and of order p^3 , except for $\langle a_1, b_1 \rangle$, which may also be the group of order p^4 defined by $a_1^{p^2} = b_1^p = [[a_1, b_1], a_1] = [[a_1, b_1], b_1] = 1$. $Z(G)$ is, in turn, a direct product of groups of order p and, possibly, one group of order p^2 . Thus G is a central product as stated. Moreover, the factor of order p^4 occurs only in case (c), the factor of order p^2 occurs if and only if $\exp Z(G) = p^2$, and a factor of order p^3 and exponent p^2 occurs only in case (b). The number of the other factors is determined by $|G|$ and $|Z(G)|$. This proves the classification part of the theorem.

For the other part, we start with the fact that, given any n such that $2n \leq p+1$, there exists a group L of maximal class and order p^{2n} , containing a maximal subgroup H with $\exp H = p$, $|H'| = p$, and on H/H' the alternating form induced by commutation is non-degenerate [6, III.14.24]. As usual, we choose generators s, s_1 for L , with $s \notin H$,

$s_1 \in H - L'$ such that, if $s_i = [s_{i-1}, s]$ then $s_i \in L_i - L_{i+1}$ [6, III.14], $i = 1, \dots, 2n - 1$. Then $H = \langle s_1, s_2, \dots, s_{2n-1} \rangle$ and defining relations for H are

$$[s_i, s_j] = s_{2n-1}^{n_{ij}}, \quad s_i^p = 1, \quad i, j = 1, \dots, 2n - 1, \quad (4)$$

for appropriate n_{ij} . Let σ be the automorphism s induces on H . Then the elements $s_i^\sigma = s_i s_{i+1}$ ($s_{2n-1}^\sigma = s_{2n-1}$) satisfy relations (4). Now let $G = H \times C$, where $C = \langle c_{2n-1}, \dots, c_p \rangle$ is elementary abelian of order p^{p-2n+1} , so $|G| = p^p$. We also denote $c_p = s_{2n-1}$; then G is defined by

$$G = \langle s_1, \dots, s_{2n-2}, c_{2n-1}, \dots, c_p \mid [s_i, s_j] = c_p^{n_{ij}}, \\ [s_i, c_j] = [c_i, c_j] = 1, s_i^p = c_j^p = 1 \rangle. \quad (5)$$

Let G_1 be defined by relations (5), except that we change the relation $s_1^p = 1$ to $s_1^p = c_p$. As we vary n from 1 to $\frac{1}{2}(p+1)$, the groups G and G_1 vary on all groups of order p^p and types (a) and (b), respectively. Let τ be the automorphism of G (or G_1) given by

$$s_i^\tau = s_i s_{i+1}, i = 1, \dots, 2n - 3; \quad s_{2n-2}^\tau = s_{2n-2} c_{2n-1}; \\ c_i^\tau = c_i c_{i+1}, i = 2n - 1, \dots, p - 1; \quad c_p^\tau = c_p. \quad (6)$$

That τ is indeed an automorphism follows from the fact that it respects relations (5). This fact is obvious for the relations $s_i^p = c_j^p = 1$ and $[s_i, c_j] = [c_i, c_j] = 1$; for the relation $s_1^p = c_p$ of G_1 it follows from $(ab)^p = a^p b^p$ which holds in all class 2 groups with $\exp G' = p$, so $(s_1 s_2)^p = s_1^p s_2^p = s_1^p$, and the relations $[s_i, s_j] = c_p^{n_{ij}}$ ($= s_{2n-1}^{n_{ij}}$) are respected by τ , because they are respected by σ , and $s_i^\tau = s_i^\sigma$ except for $i = 2n - 2$, while s_{2n-2}^σ and s_{2n-2}^τ differ from s_{2n-2} by a central element, so $[a, s_{2n-2}^\sigma] = [a, s_{2n-2}^\tau] = [a, s_{2n-2}]$ for all $a \in G$.

Since τ stabilizes a chain of subgroups of G (or G_1) it is a p -automorphism. Assume that the semi-direct product $M = G \langle \tau \rangle$ ($G_1 \langle \tau \rangle$) is regular. Then $M' = \langle s_2, \dots, c_p \rangle$ is of exponent p , so $\mathcal{U}_1(M) \subseteq Z(M)$, τ^p centralizes G , and $\tau^p = 1$. Then M is a group of order p^{p+1} and class p , hence irregular. Thus M is irregular in any case, so G and G_1 are not totally regular; i.e., groups of order p^p and of types (a), (b) are not totally regular. The total regularity of the other types will be proved later (Corollary 17).

It is convenient here to define a function $f = f(e, p)$ by means of the following table:

	$e = 1$	$e = 2$	$2 < e < \frac{1}{4}(p^2 - 5p + 12)$	$e \geq \frac{1}{4}(p^2 - 5p + 12)$
f	$\binom{p-1}{2} + 1$	$\binom{p}{2} - 1$	$e(p-4) + \frac{1}{2}(p^2 - 5p + 12)$	$e(p-2)$

THEOREM 12. *Let each section of K be totally regular, and let $p > 3$. Let $\exp K = p^e$. Then:*

- (a) $d(K) \leq p - 1$.
- (b) $|K : \mathcal{U}_1(K)| \leq p^{\binom{p-1}{2}+1}$ and $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| \leq p^{p-2}$.
- (c) $|K| \leq p^f$.

Usually more information is available when the various parameters have their extreme values. This we collect in

PROPOSITION 13. *Let K be as in Theorem 12. Then*

- (a) *If $d(K) = p - 1$, then $K_i = \mathcal{U}_{i-1}(K)$ and $[K_i, K_j] = K_{i+j}$.*
- (b) *$m(\Omega_1(K)) \leq p - 1$. If $m(\Omega_1(K)) = p - 1$, then $\mathcal{U}_1(K) \subseteq \Omega_1(K)$, and if $m(\Omega_1(K)) < p - 1$, then $|\Omega_1(K)| \leq p^{\binom{p-1}{2}}$. If $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| = p^{p-2}$, then either $\Omega_1(K) \subseteq \mathcal{U}_1(K)$ (so $|K : \mathcal{U}_1(K)| = p^{p-2}$), or $\mathcal{U}_1(K) \subseteq \Omega_1(K)$ and $|K| = p^{2p-3}$ or p^{2p-4} . If $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| = p^{p-3}$, then either $|K : \mathcal{U}_1(K)| \leq p^{p-2}$ or $|\mathcal{U}_2(K) : \mathcal{U}_3(K)| \leq p^{p-4}$.*

Theorem 12 and Proposition 13 will be proved together. The main tool is Theorem 9, supplemented by Theorem 11, so Theorem 12 and Proposition 13 hold for any group which does not involve the following groups of order p^p : elementary abelian, direct product of a cyclic group of order p^2 by $p - 2$ groups of order p , or one of the $\frac{1}{2}(p - 1)$ groups of order p^p covered by Theorem 11(a).

Proof. (a) Applying Theorem 9 to $K/\Phi(K)$ we see that $d(K) \leq p - 1$. Let $d(K) = p - 1$, then in $K^* = K/K'$ we have $|K^* : \mathcal{U}_1(K^*)| = p^{p-1}$, so (Theorem 9 again) $\mathcal{U}_1(K^*) = 1$, i.e., $\mathcal{U}_1(K) \subseteq K'$. If the inclusion is proper, we find an $N \triangleleft K$ such that $\mathcal{U}_1(K) \subseteq N \subseteq K'$ and $|K : N| = p^p$. But then K/N is of type (a) in Theorem 11, so not totally regular. Thus $\mathcal{U}_1(K) = K'$. Now induction by means of Eq. (3) yields $\mathcal{U}_{i-1}(K) = K_i$ and then $[K_i, K_j] = [\mathcal{U}_{i-1}(K), \mathcal{U}_{j-1}(K)] = \mathcal{U}_{i+j-2}(K') = \mathcal{U}_{i+j-1}(K) = K_{i+j}$.

(b) $m(\Omega_1(K)) \leq p^{p-1}$ is immediate from Theorem 9. Let N be a maximal normal abelian subgroup of $\Omega_1(K)$, and let $|N| = p^n$. Then N is self-centralizing in $\Omega_1(K)$, so $\Omega_1(K)/N$ is isomorphic to a subgroup S of $A(N)$, therefore $|S| \leq p^{\binom{n}{2}}$. Let first $n = p - 1$. Let M be an S -invariant subgroup of N of order p . Then the stability group T of the series $N \supseteq M \supseteq \{1\}$ is an elementary abelian subgroup of $A(N)$ of order p^{p-2} , and if an element $t \in \Omega_1(K)$ induces on N one of these automorphisms then $\langle N, t \rangle$ is of order p^p and type (a) in Theorem 11, an impossibility. Therefore $S \cap T = \{1\}$, so $|S| \leq p^{\binom{p-1}{2} - (p-2)}$ and $|\Omega_1(K)| = |N| |S| \leq p^{\binom{p-1}{2}+1}$. Next, if $n = p - 2$, then $|S| \leq p^{\binom{p-2}{2}}$ and $|\Omega_1(K)| \leq p^{\binom{p-1}{2}}$. Since $|\Omega_1(K)| = |K : \mathcal{U}_1(K)|$, we have proved the inequalities for this quantity.

Similarly, $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| = |\Omega_1(\mathcal{U}_1(K))| = |\Omega_1(K) \cap \mathcal{U}_1(K)|$. Denote $L = \Omega_1(K) \cap \mathcal{U}_1(K)$, then $[\Omega_1(K), \mathcal{U}_1(K)] = 1$, so L is elementary abelian, therefore $|L| \leq p^{p-1}$. Let $z \in Z(K \bmod \Omega_1(K))$, and $a \in K$, then $[z, a]^p = 1$, hence $[z, a^p] = 1$, so $[z, \mathcal{U}_1(K)] = 1$. If $|L| = p^{p-1}$, then $\langle z, L \rangle$ is abelian of order p^p at least and at least $p-1$ generators, a contradiction. Thus $|L| \leq p^{p-2}$. Moreover, if $|L| = p^{p-2}$, pick two elements outside L , $a \in \Omega_1(K)$ and $b \in \mathcal{U}_1(K)$ then again $\langle a, b, L \rangle$ is abelian with order p^p or more and $p-1$ or more generators. So we cannot find two such elements, i.e., either $L = \Omega_1(K)$ or $L = \mathcal{U}_1(K)$. Finally, if $L = \mathcal{U}_1(K)$ and we can find two elements a, b in $\Omega_1(K) - L$, independent (mod L), then again $\langle a, b, L \rangle$ violates Theorem 9 or Theorem 11(a), so now $|\Omega_1(K)| \leq p^{p-1}$, and $|\Omega_1(K)| |\mathcal{U}_1(K)| = p^{2p-4}$ or p^{2p-3} .

Finally, let $|L| = p^{p-3}$ and $|\Omega_1(K)| \geq p^{p-1}$. Let $\Omega_1(K) \supseteq N \supseteq L$ with $|N| = p^{p-1}$. Then $L \subseteq Z(N)$, so $|N'| \leq p$. If $N' = 1$, then $\mathcal{U}_1(K) \subseteq N$ and $\mathcal{U}_2(K) = 1$. Let $|N'| = p$ and $|\mathcal{U}_2(K) : \mathcal{U}_3(K)| = p^{p-3}$. Then $L \subseteq \mathcal{U}_2(K)$, hence $N' = \langle a^p \rangle$ with $a \in \mathcal{U}_1(K)$, so $\langle a, N \rangle / N'$ is elementary abelian of order p^{p-1} , or $\mathcal{U}_1(K) \subseteq \langle a, N \rangle$ and $\mathcal{U}_2(K) \subseteq N'$, a contradiction.

(c) This follows by substituting the previous bounds in $|K| = \prod_{i=1}^e |\mathcal{U}_{i-1}(K) : \mathcal{U}_i(K)|$. For $e = 1$, we have a special case of (b). Let $e > 1$. If $|K : \mathcal{U}_1(K)| \leq p^{p-2}$, we certainly have $|K| \leq p^{e(p-2)}$. If not, then by (b) we have either $|K| \leq p^{2p-3}$ and $e = 2$, or $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| \leq p^{p-3}$, and in the last case either $|K| \leq p^{p-3 + \binom{p-1}{2} + 1}$ and $e = 2$, or $|K| \leq p^{(\binom{p-1}{2} + p - 3 + (e-2)(p-4))}$. Thus $|K| \leq p^f$, where $f = \max(2p-3, \binom{p-1}{2} + 1 + p-3, e(p-2), \binom{p-1}{2} + (e-2)(p-4) + (p-3)$ and the first two terms are considered only for $e = 2$. It is easily verified that this maximum is $f(e, p)$ as defined above.

COROLLARY 14. *Let K be a 5-group, each section of which is totally regular. If $d(K) = 4$, then $|K| \leq 5^7$ and $\text{cl } K \leq 3$.*

Proof. First, by Proposition 13 we have $\Phi(K) = K' = \mathcal{U}_1(K)$, so $|\Omega_1(K)| = 5^4$. Now if $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| = 5^3$, we must have $|K| \leq 5^7$ and also $K_3 = \mathcal{U}_2(K) = 1$. So we assume $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| \leq 5^2$, and by the last claim of Proposition 13 we have $|\mathcal{U}_2(K) : \mathcal{U}_3(K)| \leq 5$, so $\mathcal{U}_2(K)$ is cyclic. Then K induces on $\mathcal{U}_2(K)$ an abelian group of automorphisms, so $[K', \mathcal{U}_2(K)] = [K', K_3] = K_5 = \mathcal{U}_4(K) = 1$ and $|K| \leq 5^8$. To improve this we apply a theorem of Blackburn [1], according to which $\mathcal{U}_1(K) = K'$ can be presented as

$$K' = \langle a, b \mid a^{p^m} = b^{p^{n+k}} = 1, [a, b] = b^{p^n}, 2k \leq m \leq n \rangle.$$

Here $K'/K'' = \mathcal{U}_1(K)/\mathcal{U}_3(K)$ is of type $(5^m, 5^n)$, and since $|K' : K''| \leq 5^3$ we must have $m \leq 1$ and $k = 0$. Thus $[a, b] = 1$, i.e., $K'' = \mathcal{U}_3(K) = 1$. This conclusion is certainly true if $|\mathcal{U}_1(K) : \mathcal{U}_2(K)| = 5$, because then $\mathcal{U}_1(K)$ is cyclic. Thus $K_4 = 1$ and $|K| \leq 5^7$.

We note also that to prove (a) of Theorem 12 and Proposition 13 we have used only the fact that factor-groups of K are totally regular, while in most of (b) we used only the total regularity of subgroups.

THEOREM 15. *Let H be not totally regular, but each proper section of H is totally regular. Let $\exp H = p^e$. Then*

(a) *H is a maximal subgroup of some minimal irregular group. In particular, H is p -abelian, and $\mathcal{U}_1(H)$ is cyclic and central in H .*

(b) *$\text{cl } H \leq 2p - 5$; if $e \geq 3$, then $\text{cl } H \leq 2p - 6$.*

(c) *$|H| \leq p^8$, where*

$$g = \binom{p-1}{2} + e, \quad e \leq 2,$$

$$= \binom{p-1}{2} + e - 1, \quad e > 2$$

(for $p \geq 5$, if $p = 3$ we have $|H| = 27$ (by (d)).

(d) *$d(H) \leq p - 2$, or $|H| = p^p$.*

Proof. (a) Let G be minimal (relative to involvement) among the groups in which H is normal and not regularly embedded. Then $G = \langle a, H \rangle$, for some a , and G is irregular. If $M \subseteq G$, then $M \cap H$ is regularly embedded in M , by minimality of G (if $H \subseteq M$) or of H , so M is regular. If $N \triangleleft G$, then similarly HN/N is regularly embedded in G/N , and G/N is regular. Thus G is a minimal irregular group. But then a^p is central in G [8], so a induces on H an automorphism α of order p , and the semi-direct product $H\langle\alpha\rangle$ is irregular (Proposition 3) and, like G , it is minimal irregular.

For the remainder of the proof, we assume that $G = H\langle\alpha\rangle$. By [8], $\exp G' = p$, and $\mathcal{U}_1(G)$ is central and cyclic. Hence the same holds for H so H is p -abelian.

(b) Let $c = \text{cl } G$. Then $\text{cl } H \leq c - 1$ [8]. If c is even, then $[G_{c/2}, G_{c/2+1}] = 1$, so a subgroup L between $G_{c/2}$ and $G_{c/2+1}$, in which the latter has index p , is elementary abelian of order at least $p^{c/2+1}$, so $c/2 + 1 \leq p - 1$ and $c \leq 2p - 4$. If $e > 2$, we can adjoin to L an element a^p of order p^2 and still have an abelian group, so $c/2 + 1 \leq p - 2$, $c \leq 2p - 6$. For odd c , we use $[G_{(c+1)/2}, G_{(c-1)/2}] \subseteq G_c$, and choose L between $G_{(c-1)/2}$ and $G_{(c+1)/2}$, with the latter of index p , then $\exp L = p$, $|L'| \leq p$, and $|L| \geq p^{(c+1)/2+1}$, so, by Theorem 11, $(c+1)/2 + 1 \leq p - 1$ and $c \leq 2p - 5$. Combining the inequalities for both cases, we have b .

For (c) and (d) we note that, by the remark following Proposition 13, the results of that proposition and of Theorem 12 hold for H , unless $|H| = p^p$. Combining the inequalities for $|H : \mathcal{U}_1(H)|$ in (b) of those results with $|\mathcal{U}_1(H)| = p^{e-1}$, we get (c). Similarly we get $d(H) \leq p-1$, and if $d(H) = p-1$, then $\mathcal{U}_1(H) = H'$ is of exponent p and cyclic, so of order p , and $|H| = p^p$.

COROLLARY 16. *If each subgroup of K can be generated by $\frac{1}{2}(p-1)$ elements, then K is totally regular. It is enough to consider subgroups of index at least $p^{(p-1)/2}$.*

Proof. If K is not totally regular, it involves a section H as in Theorem 15. Using the notations of that result, we have $H \supseteq G_{(c+1)/2}$ (or $H \supseteq L$, for even c), $G_{(c+1)/2}(L)$ is elementary abelian of order at least $p^{(c+1)/2}(p^{c/2+1})$, and $c \geq p$, so $d\{G_{(c+1)/2}\} \geq \frac{1}{2}(p+1)$ and $|H : G_{(c+1)/2}| \geq p^{(p-1)/2}$.

COROLLARY 17. *Let $|K| = p^p$. Then K is totally regular if either $(p+1)/2 < \text{cl } K < p-2$ or if K is of types (c), (d), or (e) in Theorem 11.*

Proof. If K is not totally regular, Theorem 15 implies that K is maximal in some irregular group G . Then $|G| = p^{p+1}$ and $\text{cl } G \geq p$, hence G is of maximal class. By the results of Blackburn [2, or 6.III.14], all the factors G_i/G_{i+2} , $i = 2, \dots, p-2$, have the same centralizer G_1 . Then G_1 has a central series.

$$G_1 \supset G_4 \supset G_6 \supset \dots \supset G_{p-1} \supset G_p \supset 1.$$

So $\text{cl } G_1 \leq \frac{1}{2}(p+1)$. The other maximal subgroups of G are either of maximal class or (for $H = C_G(G_{p-1})$, if $H \neq G_1$) of class $p-2$. Thus if $\text{cl } K$ is between the given limits, K cannot be maximal in such a G (of course the inequality makes sense only for $p \geq 11$). Next, any characteristic subgroup T of K is normal in G , so it is either K or some G_i . In the latter case $\exp T = p$, which eliminates types (d) and (e), in which $Z(K) \neq K$ and $\exp Z(K) = p^2$, while (c) is eliminated because $\mathcal{U}_1(K)$ and K' are distinct characteristic subgroups of the same order.

The relationship between Corollary 16 and Theorem 12(a) can be compared to that between Corollary 7 and Theorem 12(b). In both cases we have an invariant $i(K)$ of K ($\max d(H)$, $H \subseteq K$ in the first case, and $|K : \mathcal{U}_1(K)|$ in the second) and two numbers, m and n ($\frac{1}{2}(p-1)$ and $p-1$, respectively $p-2$ and $(\frac{p-1}{2} + 1)$), such that, if $i(K) \leq m$, then all sections of K are totally regular, while if each such section is totally regular, then $i(K) \leq n$.

3. MARGINAL SERIES AND PRODUCTS

A series of subgroups

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = G \quad (7)$$

with each $N_i \triangleleft G$, will be called a *p-marginal series* for G if $N_{i+1}/N_i \subseteq M(G/N_i)$, $i = 0, 1, \dots, k-1$. The *length* of (7) is k .

In particular, when $N_{i+1} = M_{i+1} = M(G/N_i)$ we obtain the *upper p-marginal series* M_i , while the *lower p-marginal series* W_i is defined by $W_0 = G$ and $W_{i+1} = \mathcal{U}_1(W_i; G)$. Thus $W_1 = \mathcal{U}_1(G)$ and, in the notation of the remark to Theorem 4, $W_i \supseteq \mathcal{U}_{(i)}(G) \supseteq \mathcal{U}_i(G)$.

It is easy to see that for any *p-marginal series* we have $W_{k-i} \subseteq N_i \subseteq M_i$, so both the upper and lower *p-marginal series* have the same length, this being the shortest length of all *p-marginal series* of G .

From now on let G be any group. Let V be any variety of groups. A series (7) will be termed a *V-marginal series* for G if $N_{i+1}/N_i \subseteq M_V(G/N_i)$. We define the upper and lower *V-marginal series* as before, replacing $M(H)$ by $M_V(H)$ and $\mathcal{U}_1(N; H)$ by the subgroup $V(N; H)$ generated by all elements $v_\alpha(a_1, \dots, a_1 b, \dots, a_n) v_\alpha(a_1, \dots, a_i, \dots, a_n)^{-1}$, for all $i = 1, \dots, n$, all $a_i \in H$, $b \in N$ and all laws $v_\alpha = v_\alpha(x_1, \dots, x_n)$ of V (or of some set of laws defining V).

A group possessing a *V-marginal series* may be termed *V-nilpotent*, its *V-class* being the shortest length of such a series, which is the common length of both the upper and lower *V-marginal series*.

THEOREM 18. *The class of V-nilpotent groups of V-class at most k is a variety V_k . Given any group G, the corresponding verbal subgroup of G, $V_k(G)$, is the kth term of the lower V-marginal series, and the marginal subgroup $M_{(V_k)}(G)$ is contained in the kth term $M_{V_k}(G)$ of the upper V-marginal series of G.*

These definitions and assertions can be generalized. Let U be another variety.

DEFINITION. The *marginal product* UmV of U and V is the class of all groups G such that $G/M_V(G) \in U$.

THEOREM 19. *The class UmV is a variety, the corresponding verbal subgroup being $U(V(G); G)$ and the corresponding marginal subgroup being contained in the subgroup $M_V(G \bmod M_U(G))$.*

Theorem 18 follows by induction from Theorem 19.

Proof. Let $\{v_\alpha\}$ and $\{u_\beta\}$ be sets of laws defining the varieties V and U . It is easy to see that UmV is defined by all laws of the form

$$u_\beta(x_1, \dots, x_i v_\alpha(y_1, \dots, y_n), \dots, x_m) u_\beta(x_1, \dots, x_i, \dots, x_m)^{-1}. \quad (8)$$

This proves the first two assertions. Next, if $z \in M_{UmV}(G)$, let

$$v_\alpha(y_1, \dots, y_j z, \dots, y_n) = v_\alpha(y_1, \dots, y_j, \dots, y_n) w,$$

substitute $y_j z$ for y_j in (8) and equate to 1. This yields $w \in M_U(G)$, i.e., $z \in M_V(G \bmod M_U(G))$.

It follows from (8) that if V and U are finitely based, so is UmV .

The assertion on the marginal subgroups implies that, for a third variety W , one has $(UmV)mW \subseteq Um(VmW)$. It is not clear if equality holds here, although it certainly holds in many cases (we certainly have equality if there is equality between the marginal subgroups in Theorem 19). The inclusion implies that the class of V -nilpotent groups of V -class at most k is the largest among the k times iterated marginal products of V by itself.

A further problem (raised by E. Rips) is whether the marginal product is cancellative.

Notes added in Proof. 1. The results of the last section have been noted previously, in C. R. LEEDHAM-GREEN and MCKAY, *Bauer-invariants, etc.*, *Acta Math.* **137** (1976), 99–150.

2. The 5-groups, all sections of which are totally regular, have by now been almost completely classified.

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